

# GENERALIZATIONS OF KAPLANSKY THEOREM RELATED TO LINEAR OPERATORS

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**ABSTRACT.** The purpose of this paper is to generalize a very famous result on products of normal operators, due to I. Kaplansky. The context of generalization is that of bounded hyponormal and unbounded normal operators on complex separable Hilbert spaces. Some examples "spice up" the paper.

## 1. INTRODUCTION

Normal operators are a major class of bounded and unbounded operators. Among their virtues, they are the largest class of single operators for which the spectral theorem is proved (cf. [15]). There are other classes of interesting non-normal operators such as hyponormal and subnormal operators (among others). They have been of interest to many mathematicians and have been extensively investigated enough so that even monographs have been devoted to them. See for instance [3] and [10].

In this paper we are mainly interested in generalizing the following result to unbounded normal and bounded hyponormal operators:

**Theorem 1.1** (Kaplansky, [8]). *Let  $A$  and  $B$  be two bounded operators on a Hilbert space such that  $AB$  and  $A$  are normal. Then  $B$  commutes with  $AA^*$  iff  $BA$  is normal.*

Before recalling some essential background, we make the following observation:

*All operators are linear and are defined on a separable complex Hilbert space, which we will denote henceforth by  $H$ .*

A bounded operator  $A$  on  $H$  is said to be normal if  $AA^* = A^*A$ .  $A$  is called hyponormal if  $AA^* \leq A^*A$ , that is iff  $\|A^*x\| \leq \|Ax\|$  for all  $x \in H$ . Hence a normal operator is always hyponormal. Obviously, a hyponormal operator need not be normal. However, and in a finite-dimensional setting, a hyponormal operator is normal too. This is proved via a nice and simple trace argument (see e.g. [7]).

Since the paper is also concerned with unbounded operators, and for the readers convenience, we recall some known notions and results about unbounded operators.

If  $A$  and  $B$  are two unbounded operators with domains  $D(A)$  and  $D(B)$  respectively, then  $B$  is said to be an extension of  $A$ , and we denote it by  $A \subset B$ , if  $D(A) \subset D(B)$  and  $A$  and  $B$  coincide on each element of  $D(A)$ . An operator  $A$  is said to be densely defined if  $D(A)$  is dense in  $H$ . The (Hilbert) adjoint of  $A$  is denoted by  $A^*$  and it is known to be unique if  $A$  is densely defined. An operator  $A$

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is said to be closed if its graph is closed in  $H \times H$ . We say that the unbounded  $A$  is self-adjoint if  $A = A^*$ , and we say that it is normal if  $A$  is closed and  $AA^* = A^*A$ . Recall also that the product  $BA$  is closed if for instance  $B$  is closed and  $A$  is bounded, and that if  $A$ ,  $B$  and  $AB$  are densely-defined, then only  $B^*A^* \subset (AB)^*$  holds; and if further  $A$  is assumed to be bounded, then  $B^*A^* = (AB)^*$ .

The notion of hyponormality extends naturally to unbounded operators. an unbounded  $A$  is called hyponormal if:

- (1)  $D(A) \subset D(A^*)$ ,
- (2)  $\|A^*x\| \leq \|Ax\|$  for all  $x \in D(A)$ .

It is also convenient to recall the following theorem which appeared in [16], but we state it in the form we need.

**Theorem 1.2** (Stochel). *If  $T$  is a closed subnormal (resp. closed hyponormal) operator and  $S$  is a closed hyponormal (resp. closed subnormal) operator verifying  $XT^* \subset SX$  where  $X$  is a bounded operator, then both  $S$  and  $T^*$  are normal once  $\ker X = \ker X^* = \{0\}$ .*

Any other result or notion (such as the classical Fuglede-Putnam theorem, the polar decomposition, subnormality etc...) will be assumed to be known by readers. For more details, the interested reader is referred to [1], [2], [6], [14] and [15]. For other works related to products of normal (bounded and unbounded) operators, the reader may consult [5], [9], [11], [12] and [13], and the references therein.

## 2. MAIN RESULTS: THE BOUNDED CASE

The following known lemma is essential (we include a proof):

**Lemma 2.1.** *Let  $S$  and  $T$  be two bounded self-adjoint operators on a Hilbert space  $H$ . If  $U$  is any operator, then*

$$S \geq T \implies USU^* \geq UTU^*.$$

*Proof.* Let  $x \in H$ . We have

$$\begin{aligned} \langle USU^*x, x \rangle &= \langle SU^*x, U^*x \rangle \\ &\geq \langle TU^*x, U^*x \rangle \\ &= \langle UTU^*x, x \rangle. \end{aligned}$$

□

As a direct application of the previous result we have the following Kaplansky-like theorem:

**Proposition 2.1.** *Let  $A$  and  $B$  be two bounded operators on a Hilbert space such that  $A$  is normal and  $AB$  is hyponormal. Then*

$$AA^*B = BAA^* \implies BA \text{ is hyponormal.}$$

*Proof.* Since  $A$  is normal, we know that

$$A = PU = UP$$

where  $P$  is positive and  $U$  is unitary. Hence

$$AA^*B = BAA^* \implies P^2B = BP^2 \implies PB = BB$$

so that

$$U^*ABU = U^*UPBU = PBU = BA.$$

Finally, we have

$$\begin{aligned} BA(BA)^* &= (U^*(AB)U)(U^*ABU)^* \\ &= U^*ABUU^*(AB)^*U \\ &= U^*(AB)(AB)^*U \\ &\leq U^*(AB)^*ABU \\ &= (BA)^*BA. \end{aligned}$$

□

The reverse implication does not hold in the previous result (even if  $A$  is self-adjoint) as shown in the following example:

**Example 1.** Let  $A$  and  $B$  be acting on the standard basis  $(e_n)$  of  $\ell^2(\mathbb{N})$  by:

$$Ae_n = \alpha_n e_n \text{ and } Be_n = e_{n+1}, \forall n \geq 1$$

respectively. Assume further that  $\alpha_n$  is bounded, *real-valued* and *positive*, for all  $n$ . Hence  $A$  is self-adjoint (hence normal!) and positive. Then

$$ABe_n = \alpha_n e_{n+1}, \forall n \geq 1.$$

For convenience, let us carry out the calculations as infinite matrices. Then

$$AB = \begin{bmatrix} 0 & 0 & & & 0 \\ \alpha_1 & 0 & 0 & & \\ 0 & \alpha_2 & 0 & 0 & \\ & 0 & \alpha_3 & 0 & \ddots \\ & & 0 & \ddots & 0 \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix} \text{ so that } (AB)^* = \begin{bmatrix} 0 & \alpha_1 & & & 0 \\ 0 & 0 & \alpha_2 & & \\ 0 & 0 & 0 & \alpha_3 & \\ & 0 & 0 & 0 & \ddots \\ & & 0 & \ddots & 0 \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Hence

$$AB(AB)^* = \begin{bmatrix} 0 & 0 & & & 0 \\ 0 & \alpha_1^2 & 0 & & \\ 0 & 0 & \alpha_2^2 & 0 & \\ & 0 & 0 & \alpha_3^2 & \ddots \\ & & 0 & \ddots & \ddots \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$(AB)^*AB = \begin{bmatrix} \alpha_1^2 & 0 & & & 0 \\ 0 & \alpha_2^2 & 0 & & \\ 0 & 0 & \alpha_3^2 & 0 & \\ & 0 & 0 & \ddots & \ddots \\ & & 0 & \ddots & \ddots \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

It thus becomes clear that  $AB$  is hyponormal iff  $\alpha_n \leq \alpha_{n+1}$ .

Similarly

$$BAe_n = \alpha_{n+1}e_{n+1}, \forall n \geq 1.$$

Whence the matrix representing  $BA$  is given by:

$$BA = \begin{bmatrix} 0 & 0 & & & 0 \\ \alpha_2 & 0 & 0 & & \\ 0 & \alpha_3 & 0 & 0 & \\ & 0 & \alpha_4 & 0 & \ddots \\ & & 0 & \ddots & 0 \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix} \quad \text{so that } (BA)^* = \begin{bmatrix} 0 & \alpha_2 & & & 0 \\ 0 & 0 & \alpha_3 & & \\ 0 & 0 & 0 & \alpha_4 & \\ & 0 & 0 & 0 & \ddots \\ & & 0 & \ddots & 0 \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Therefore,

$$BA(BA)^* = \begin{bmatrix} 0 & 0 & & & 0 \\ 0 & \alpha_2^2 & 0 & & \\ 0 & 0 & \alpha_3^2 & 0 & \\ & 0 & 0 & \alpha_4^2 & \ddots \\ & & 0 & \ddots & \ddots \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$(BA)^*BA = \begin{bmatrix} \alpha_2^2 & 0 & & & 0 \\ 0 & \alpha_3^2 & 0 & & \\ 0 & 0 & \alpha_4^2 & 0 & \\ & 0 & 0 & \ddots & \ddots \\ & & 0 & \ddots & \ddots \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

Accordingly,  $BA$  is hyponormal iff  $\alpha_n \leq \alpha_{n+1}$  (thankfully, this is the same condition for the hyponormality of  $AB$ ).

Finally,

$$BA^2 = \begin{bmatrix} 0 & 0 & & & 0 \\ \alpha_1^2 & 0 & 0 & & \\ 0 & \alpha_2^2 & 0 & 0 & \\ & 0 & \alpha_3^2 & 0 & \ddots \\ & & 0 & \ddots & 0 \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix} \neq A^2B = \begin{bmatrix} 0 & 0 & & & 0 \\ \alpha_2^2 & 0 & 0 & & \\ 0 & \alpha_3^2 & 0 & 0 & \\ & 0 & \alpha_4^2 & 0 & \ddots \\ & & 0 & \ddots & 0 \\ 0 & & & \ddots & \ddots & \ddots \end{bmatrix}$$

*Remark.* An explicit example of such an  $(\alpha_n)$  verifying the required hypotheses would be to take:

$$\begin{cases} \alpha_1 = 0 \\ \alpha_{n+1} = \sqrt{2 + \alpha_n} \end{cases}$$

Then  $(\alpha_n)$  is bounded (in fact,  $0 \leq \alpha_n < 2$ , for all  $n$ ), increasing and such that  $\alpha_1 = 0 \neq \alpha_2 = \sqrt{2}$ .

Going back to Proposition 2.1, we observe that the result obviously holds by replacing "hyponormal" by "co-hyponormal". Thus we have

**Proposition 2.2.** *Let  $A$  and  $B$  be two bounded operators on a Hilbert space such that  $A$  is normal and  $AB$  is co-hyponormal. Then*

$$AA^*B = BAA^* \implies BA \text{ is co-hyponormal.}$$

*Remark.* The same previous example, mutatis mutandis, works as a counterexample to show that " $BA$  co-hyponormal  $\Rightarrow AA^*B = BAA^*$ " need not hold.

We now come to a very important result of the paper:

**Theorem 2.1.** *Let  $A, B$  be two bounded operators such that  $A$  is also normal. Assume that  $AB$  is hyponormal and that  $BA$  is co-hyponormal. Then*

$$AA^*B = BA^*A \iff BA \text{ and } AB \text{ are normal.}$$

*Proof.*

- (1) " $\implies$ ": Since  $A$  is normal, we have  $A = UP = PU$  where  $U$  is unitary and  $P$  is positive. Since  $AA^*B = BA^*A$ , we obtain  $P^2B = BP^2$  or just  $BP = PB$  by the positivity of  $P$ .

Therefore, we may write

$$\begin{aligned} (AB)^*AB &= (U(BA)U^*)^*(U(BA)U^*) \\ &= U(BA)^*U^*U(BA)U^* \\ &= U(BA)^*(BA)U^* \\ &\leq U(BA)(BA)^*U^* \text{ (since } BA \text{ is co-hyponormal and by Lemma 2.1)} \\ &= U(BA)U^*U(BA)^*U^* \\ &= (AB)(AB)^*, \end{aligned}$$

that is  $AB$  is co-hyponormal. Since it is already hyponormal, it immediately follows that  $AB$  is normal.

To prove that  $BA$  is normal we apply a similar idea and we have

$$\begin{aligned} (BA)(BA)^* &= (U^*(AB)U)(U^*(AB)U)^* \\ &= U^*(AB)^*UU^*(AB)^*U \\ &= U^*(AB)(AB)^*U \\ &\leq U^*(AB)^*(AB)U \text{ (since } AB \text{ is hyponormal and by Lemma 2.1)} \\ &= U^*(AB)^*UU^*(AB)U \\ &= (BA)^*(BA), \end{aligned}$$

that is  $BA$  is hyponormal, and since it is also co-hyponormal, we conclude that  $BA$  is normal.

- (2) " $\impliedby$ ": To prove the reverse implication, we use the celebrated Fuglede-Putnam theorem (see e.g. [2]) and we have:

$$\begin{aligned} ABA &= ABA \implies A(BA) = (AB)A \\ &\implies A(BA)^* = (AB)^*A \\ &\implies AA^*B^* = B^*A^*A \\ &\implies BAA^* = A^*AB. \end{aligned}$$

This completes the proof.

□

### 3. MAIN RESULTS: THE UNBOUNDED CASE

We start this section by giving a counterexample that shows that the same assumptions, as in Theorem 1.1, would not yield the same results if  $B$  is an unbounded operator, let alone the case where both  $A$  and  $B$  are unbounded.

What we want is a normal bounded operator  $A$  and an unbounded (and closed) operator  $B$  such that  $BA$  is normal,  $A^*AB \subset BA^*A$  but  $AB$  is not normal.

**Example 2.** Let

$$Bf(x) = e^{x^2}f(x) \text{ and } Af(x) = e^{-x^2}f(x)$$

on their respective domains

$$D(B) = \{f \in L^2(\mathbb{R}) : e^{x^2}f \in L^2(\mathbb{R})\} \text{ and } D(A) = L^2(\mathbb{R}).$$

Then it is well known that  $A$  is bounded and self-adjoint (hence normal), and that  $B$  is self-adjoint (hence closed).

Now  $AB$  is not normal for it is not closed as  $AB \subset I$ .  $BA$  is normal as  $BA = I$  (on  $L^2(\mathbb{R})$ ). Hence  $AB \subset BA$  which implies that

$$AAB \subset ABA \implies AAB \subset ABA \subset BAA.$$

Now, we state and prove the generalization of Theorem 1.1 to unbounded operators. We have

**Theorem 3.1.** *Let  $B$  be an unbounded closed operator and  $A$  a bounded one such that  $AB$  and  $A$  are normal. Then*

$$BA \text{ normal} \implies A^*AB \subset BA^*A.$$

*If further  $BA$  is hyponormal (resp. subnormal), then*

$$BA \text{ normal} \iff A^*AB \subset BA^*A.$$

*Proof.*

- (1) " $\implies$ ": Since  $AB$  and  $BA$  are normal, the equation

$$A(BA) = (AB)A$$

implies that

$$A(BA)^* = (AB)^*A$$

by the Fuglede-Putnam theorem (see e.g. [2]). Hence

$$AA^*B^* \subset B^*A^*A \text{ or } A^*AB \subset BA^*A.$$

- (2) " $\impliedby$ ": The idea of proof in this case is similar in core to Kaplansky's (cf. [8]). Let  $A = UP$  be the polar decomposition of  $A$ , where  $U$  is unitary and  $P$  is positive (remember that they also commute and that  $P = \sqrt{A^*A}$ ), then one may write

$$U^*ABU = U^*UPBU = PBU \subset BPU = BA$$

or

$$U^*AB = U^*\overline{AB} = U^*((AB)^*)^* \subset BAU^*$$

(by the closedness of  $AB$ ). Since  $(AB)^*$  is normal, it is closed and subnormal. Since  $B$  is closed and  $A$  is bounded,  $BA$  is closed. Since it is

hyponormal, Theorem 1.2 applies and yields the normality of  $BA$  as  $U$  is invertible.

The proof is very much alike in the case of subnormality. □

Imposing another commutativity condition allows us to generalize Theorem 1.1 to unbounded normal operators by bypassing hyponormality and subnormality:

**Theorem 3.2.** *Let  $B$  be an unbounded closed operator and  $A$  a bounded one such that all of  $AB$ ,  $A$  and  $B$  are all normal. Then*

$$A^*AB \subset BA^*A \text{ and } AB^*B \subset B^*BA \implies BA \text{ normal}.$$

The proof is partly based on the following interesting result of maximality of self-adjoint operators:

**Proposition 3.1** (Devinatz-Nussbaum-von Neumann, [4]). *Let  $A$ ,  $B$  and  $C$  be unbounded self-adjoint operators. Then*

$$A \subseteq BC \implies A = BC.$$

Now we give the proof of Theorem 3.2:

*Proof.* First,  $BA$  is closed as  $A$  is bounded and  $B$  is closed. So  $BA(BA)^*$  (and  $(BA)^*BA$ ) is self-adjoint. Then we have

$$A^*ABB^* \subset BA^*AB^* = BAA^*B^* \subset BA(BA)^*$$

and hence

$$BA(BA)^* \subset (A^*ABB^*)^* = BB^*A^*A$$

so that Proposition 3.1 gives us

$$BA(BA)^* = BB^*A^*A$$

for both  $BB^*$  and  $A^*A$  are self-adjoint since  $B$  is closed and  $A$  is bounded respectively.

Similarly

$$A^*AB^*B \subset A^*B^*BA \subset (BA)^*BA.$$

Adjoining the previous "inclusion" and applying again Proposition 3.1 yield

$$(BA)^*BA = B^*BA^*A = BB^*A^*A,$$

establishing the normality of  $BA$ . □

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